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Saddle points criteria in nondifferentiable multiobjective programming with V-invex functions via an η-approximation method

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ABSTRACT

The η-approximation method is used to a characterization of solvability of nonconvex nondifferentiable multiobjective programming problems. A family of η-approximated vector optimization problems is constructed in this approach for the original nondifferentiable multiobjective programming problem. The definitions of a vector-valued η-Lagrange function and of an η-saddle point for this family of η-approximated vector optimization problems are introduced. Thus, the equivalence between a (weak) Pareto optimum of the original multiobjective programming problems and an η-saddle point of the η-Lagrange function in its associated η-approximated vector optimization problems is established under V-invexity.

1. Introduction

In this paper, we consider the following multiobjective programming problem

$$\begin{aligned} &V\text{-Minimize } f(x) := (f_1(x), \ldots, f_k(x)) \\ &\text{subject to } g(x) := (g_1(x), \ldots, g_m(x)) \leqq 0, \end{aligned} \tag{VP}$$

where $f_i : X \to R,\ i = 1, \ldots, k$, and $g_j : X \to R,\ j = 1, \ldots, m$, are locally Lipschitz functions on a nonempty open set $X \subset R^n$. We call (VP) the original multiobjective programming problem.

Let

$$D := \left\{x \in X : g_j(x) \leqq 0,\ j = 1, \ldots, m\right\}$$

denote the set of all feasible solutions in the original multiobjective programming problem (VP). Further, we denote by

$$J(x) := \left\{i \in J : g_j(x) = 0\right\}$$

the index set of all active constraints of (VP) at an arbitrary feasible solution x.

The following convention for equalities and inequalities will be used throughout the paper.

For any $x = (x_1, x_2, \ldots, x_n)^T,\ y = (y_1, y_2, \ldots, y_n)^T$, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \ldots, n$;
(ii) $x < y$ if and only if $x_i < y_i$ for all $i = 1, 2, \ldots, n$;
(iii) $x \leqq y$ if and only if $x_i \leq y_i$ for all $i = 1, 2, \ldots, n$;
(iv) $x \leq y$ if and only if $x \leqq y$ and $x \neq y$.

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Note here that the symbol "V-Minimize" stands for vector minimization—thus a weak Pareto optimal solution or Pareto optimal solution in the following sense:

Definition 1. A feasible point $\bar{x}$ is said to be a Pareto optimal solution (efficient solution) for (VP) if and only if there exists no $x \in D$ such that

$$f(x) \leq f(\bar{x}).$$

Definition 2. A feasible point $\bar{x}$ is said to be a weak Pareto optimal solution (weakly efficient solution, weak minimum) for (VP) if and only if there exists no $x \in D$ such that

$$f(x) < f(\bar{x}).$$

In most real-life problems, decisions are made taking into account several conflicting criteria, rather than by optimizing a single objective. Such a problem is called a multiobjective programming problem when both the criteria and the constraints that determine the feasible set of alternatives can be mathematically expressed by functions. Many different approaches have been designed to characterize solvability of such optimization problems. One of them is using saddle points criteria.

Saddle points criteria are very important in optimization theory. Due to its wide application in multiobjective programming problems, saddle points criteria are more and more investigated by many authors (see, for example, [1–13], and others).

The main purpose of this paper is to present a transformation method for a new class of nonconvex nondifferentiable multiobjective programming problems. The method used in the derivation of optimality criteria is based on the so-called η-approximation of all functions constituting the original vector optimization problem. This approach, named the η-approximation method, was introduced by Antczak [14] for differentiable multiobjective programming problems with invex functions (with respect to the same function η). It allows to convert the original multiobjective programming problem into an another equivalent vector optimization problem named an η-approximated optimization problem. An η-approximation vector optimization problem is constructed by a modification of both the objective and constraints functions in the original multiobjective programming problem at an arbitrary but fixed feasible point $\bar{x}$.

In [15], Antczak extended the η-approximation method to the nonsmooth case. In this case, for the original nonsmooth multiobjective programming problem, a family of its associated vector optimization problems is constructed by modifying both the objectives and the constraints at an arbitrary but fixed point $\left(\bar{x}, \bar{\xi}, \bar{\zeta}\right)$, where $\bar{x}$ is a feasible solution for the considered vector optimization problem, and $\bar{\xi} = \left(\bar{\xi}_1, \ldots, \bar{\xi}_k\right)$, $\bar{\zeta} = \left(\bar{\zeta}_1, \ldots, \bar{\zeta}_m\right)$, $\bar{\xi}_i$, $i = 1, \ldots, k$, $\bar{\zeta}_j$, $j = 1, \ldots, m$, are Clarke's generalized gradients of the objective functions f_i, $i = 1, \ldots, k$, and the constraint functions g_j, $j = 1, \ldots, m$, at $\bar{x}$, respectively. This construction depends heavily on results proved in this paper, which connect the (weakly) efficient points of the original nondifferentiable multiobjective programming problem to the (weakly) efficient points of the modified vector minimization problem belonging to the above described family of vector optimization problems. This equivalence between these vector optimization problems is established under assumption that all functions constituting the original multiobjective programming problem are nondifferentiable V-invex at $\bar{x}$ with respect to the same function η and with respect to, not necessarily, the same function α on the set of all feasible solutions in the original multiobjective programming problem. In this way, we obtain a family of the so-called associated η-approximated vector optimization problems with the same (weak) Pareto optimal solution $\bar{x}$ and the same optimal value as in the original nonsmooth multiobjective programming problem. Furthermore, the associated η-approximated vector optimization problems have, in general, simpler forms than the original nonlinear nonsmooth multiobjective programming problem and, therefore, they are easier to solve (in most cases, they are smooth and linear (or convex)).

In this paper, we develop new saddle points criteria for nonconvex nondifferentiable multiobjective programming problems with inequality constraints. With the aid of the so-called η-saddle point criteria defined for the η-Lagrange function defined in an η-approximated vector optimization problem, we characterize solvability of a new class of nonconvex nondifferentiable multiobjective programming problems. Namely, the equivalence between the original multiobjective programming problem and its associated η-approximated vector optimization problems is established under nondifferentiable V-invexity. We show that a (weak) Pareto optimal solution of the original multiobjective programming problem and a so-called η-saddle point of the vector-valued η-Lagrange function defined for its associated η-approximated vector optimization problems in this method are equivalent.

This paper not only generalizes the results obtained in [16], but also expands considerably the class of nonconvex vector optimization problems for which the equivalence between (weak) Pareto optimality and an η-saddle point of the η-Lagrange function can be guaranteed. In other words, the applicability of the η-approximation method, previously used in the case of differentiable vector optimization problems with invex functions, is extended to a new class of nonsmooth vector optimization problems.

2. Preliminaries

In this section we introduce some notions and definitions.

Definition 3. A real-valued function $f : X \to R$ is said to be locally Lipschitz on X if, for any $x \in X$, there exist a neighborhood U of x and a positive constant $K_x > 0$ such that, for every $y, z \in U$,

$$|f(y) - f(z)| \leqq K_x \|y - z\| .$$

Definition 4. [17] If $f : R^n \to R$ is a locally Lipschitz function at $x \in R^n$, the generalized derivative (in the sense of Clarke) of f at $x \in R^n$ in the direction $v \in R^n$, denoted $f^0(x; v)$, is given by

$$f^0(x; v) = \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

Definition 5. [17] The Clarke's generalized gradient of f at $x \in R^n$, denoted $\partial f(x)$, is defined as follows:

$$\partial f(x) = \left\{ \xi \in R^n : f^0(x; v) \geq \xi^T v \text{ for all } v \in R^n \right\}. \tag{1}$$

Remark 6. It follows that, for any $v \in R^n$,

$$f^0(x; v) = \max \left\{ \xi^T v : \xi \in \partial f(x) \right\}.$$

In this section, in terms of the Clarke subdifferential, we give a definition of nondifferentiable V-invex functions introduced by Jeyakumar and Mond [18].

Definition 7. Let $f = (f_1, \dots, f_k) : X \to R^k$ be defined on a nonempty subset X of R^n, where f_i, $i = 1, \dots, k$, are locally Lipschitz. A vector function $f : X \to R^k$ is said to be (nondifferentiable) V-invex at $u \in X$ on X (with respect to η and $\alpha := (\alpha_1, \dots, \alpha_k)$) if, there exist functions $\eta : X \times X \to R^n$ and $\alpha_i : X \times X \to R_+ \setminus \{0\}$, $i = 1, \dots, k$, such that, for all $x \in X$, the following inequalities

$$f_i(x) - f_i(u) \geqq \alpha_i(x, u) \xi_i^T \eta(x, u), \quad i = 1, \dots, k, \tag{2}$$

hold for each $\xi_i \in \partial f_i(u)$, $i = 1, \dots, k$. If the inequalities (2) are satisfied for any $u \in X$, then f is (nondifferentiable) V-invex (with respect to η and α) on X. Each function f_i satisfying (2) at any $u \in X$ is said to be (locally Lipschitz) α_i-invex with respect to η on X.

We now define a vector-valued Lagrange function for the original multiobjective programming problem (VP) as follows

$$L(x, \lambda, \mu) := \operatorname{diag} \lambda f(x) + \frac{1}{k}\mu^T g(x) e = \left(\lambda_1 f_1(x) + \frac{1}{k}\mu^T g(x), \dots, \lambda_k f_k(x) + \frac{1}{k}\mu^T g(x) \right), \tag{3}$$

where $\lambda \in R^k_+$, $\mu \in R^m_+$, $e = (1, \dots, 1) \in R^k$ and

$$\operatorname{diag} \lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix}.$$

It is well known (see, for example, [19–23]), that the Karush–Kuhn–Tucker conditions are necessary for optimality in such vector optimization problems under assumption that a suitable constraint qualification is satisfied.

In the paper, we use the following constraint qualification:

At a point $\bar{x} \in D$, let us define

$$\Omega(\bar{x}) := \begin{cases} \left\{ v \in R^n : g_j^0(\bar{x}; v) < 0 \text{ for any } j \in J(\bar{x}) \right\} & \text{if } J(\bar{x}) \neq \varnothing \\ R^n & \text{if } J(\bar{x}) = \varnothing. \end{cases}$$

Constraint Qualification (CQ): At a point $\bar{x} \in D$, it holds that $\Omega(\bar{x}) \neq \varnothing$.

Theorem 8. *Let $\bar{x}$ be a (weak) Pareto optimal solution in problem (VP) and Constraint Qualification (CQ) be satisfied at $\bar{x}$. Then there exist $\bar{\lambda} \in R^k$ and $\bar{\mu} \in R^m$ such that*

$$0 \in \sum_{i=1}^{k} \bar{\lambda}_i \partial f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \partial g_j(\bar{x}), \tag{4}$$

$$\bar{\mu}_j g_j(\bar{x}) = 0, \quad j = 1, \dots, m, \tag{5}$$

$$\bar{\lambda} \geq 0, \qquad \bar{\mu} \geqq 0. \tag{6}$$

For the considered nonlinear multiobjective programming problem (VP), it can be formulated also the so-called Generalized Slater's Constraint Qualification (GSCQ). This constraint qualification has been introduced in [15] as follows:

Generalized Slater's Constraint Qualification (GSCQ): For problem (VP), assume that there exists a point $\widetilde{x} \in D$ such that $g_j(\widetilde{x}) < 0,\ j \in J(\overline{x})$, and, moreover, the constraint functions $g_j,\ j \in J(\overline{x})$, are β_j-invex at $\overline{x}$ on D with respect to the same function η.

Remark 9. It is not difficult to see that if we assume the following Generalized Slater's Constraint Qualification (GSCQ), then, at the point $\overline{x} \in D$, it holds that $\Omega(\overline{x}) \neq \varnothing$. However, as it follows from the formulation of (GSCQ), the constraint functions $g_j,\ j \in J(\overline{x})$, should be assumed to be β_j-invex at $\overline{x}$ on D with respect to the same function η.

3. A family of η-approximated vector optimization problems and optimality conditions

Let $\overline{x}$ be the given feasible solution in the original nonsmooth multiobjective programming problem (VP) and, moreover, $\overline{\xi} = \left(\overline{\xi}_1, \ldots, \overline{\xi}_k\right),\ \overline{\zeta} = \left(\overline{\zeta}_1, \ldots, \overline{\zeta}_m\right),\ \overline{\xi}_i,\ i = 1, \ldots, k,\ \overline{\zeta}_j, j = 1, \ldots, m$, are Clarke's generalized gradients of f_i, $i = 1, \ldots, k$, and $g_j, j = 1, \ldots, m$, at $\overline{x}$, respectively, that is, $\overline{\xi}_i \in \partial f_i(\overline{x}),\ i = 1, \ldots, k,\ \overline{\zeta}_j \in \partial g_j(\overline{x}),\ j = 1, \ldots, m$, respectively. We consider a family of the following η-approximated vector optimization problems $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$ given by

$$\begin{aligned} &V\text{-Minimize } \left(f_1(\overline{x}) + \overline{\xi}_1^T \eta(x, \overline{x}), \ldots, f_k(\overline{x}) + \overline{\xi}_k^T \eta(x, \overline{x})\right) \\ &\text{subject to } g_j(\overline{x}) + \overline{\zeta}_j^T \eta(x, \overline{x}) \leqq 0, \quad j = 1, \ldots, m, \end{aligned} \qquad (VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$$

where f, g, X are defined as in the original multiobjective programming problem (VP) and η is a vector-valued function defined by $\eta : X \times X \to R^n$. Throughout the paper, we will assume that η satisfies the following condition: $\eta(x, \overline{x}) \neq 0$ for any $x \in X$ such that $x \neq \overline{x}$.

Let

$$D\left(\overline{x}, \overline{\zeta}\right) := \left\{x \in X : g_j(\overline{x}) + \overline{\zeta}_j^T \eta(x, \overline{x}) \leqq 0,\ j = 1, \ldots, m\right\}$$

denote the set of all feasible solutions of $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$.

Antczak [15] established the equivalence between the original multiobjective programming problem (VP) and its η-approximated vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$ belonging to a family of described above vector optimization problems in the following sense: if $\overline{x}$ is a (weak) Pareto optimal solution in (VP), then it is also (weak) efficient in its η-approximated vector optimization problem, and also conversely, if $\overline{x}$ is a (weak) Pareto optimal point in its η-approximated vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$, then it is also (weak) efficient in (VP).

In this paper, we prove the equivalence between the original multiobjective programming problem (VP) and its η-approximated vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$ in a slightly different way. We introduce the so-called η-saddle point criteria for η-approximated vector optimization problems $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$ and then we use them to characterize solvability of the original nonsmooth multiobjective programming problem (VP).

Now, we establish, under the nondifferentiable V-invexity assumption imposed on the constraint function, that the set of all feasible solutions in the original multiobjective programming problem (VP) is contained in the set of all feasible solutions in its η-approximated vector optimization problems $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$.

Lemma 10. *Let $\overline{x}$ be an arbitrary given feasible point in (VP) such that $g(\overline{x}) = 0$. Further, assume that the constraint function g is nondifferentiable V-invex with respect to η at $\overline{x}$ on D. Then any feasible solution in (VP) is also feasible in $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$, that is, $D \subset D(\overline{x})$.*

Proof. By assumption, $\overline{x}$ be an arbitrary given feasible point in (VP). By assumption, g is a nondifferentiable V-invex function at $\overline{x}$ on D with respect to η. Hence, by Definition 7, the following inequalities

$$g_j(x) - g_j(\overline{x}) \geqq \beta_j(x, \overline{x})\, \overline{\zeta}_j^T \eta(x, \overline{x}) \qquad (7)$$

hold for all $x \in D$ and any $\overline{\zeta}_j \in \partial g_j(\overline{x}),\ j \in J$. From the assumption, for any solution x feasible in (VP), it follows that

$$0 = g(\overline{x}) \geqq g(x). \qquad (8)$$

Thus, by (7) and (8),

$$\beta_j(x, \overline{x})\, \overline{\zeta}_j^T \eta(x, \overline{x}) \leqq 0.$$

Since $\beta_j(x, \overline{x}) > 0,\ j \in J$, for all $x \in D$, then the inequalities

$$\overline{\zeta}_j^T \eta(x, \overline{x}) \leqq 0. \qquad (9)$$

hold for all $x \in D$ and any $\overline{\zeta}_j \in \partial g_j(\overline{x}),\ j \in J$. Therefore, (9) yields, for any $j \in J$,

$$g_j(\overline{x}) + \overline{\zeta}_j^T \eta(x, \overline{x}) \leqq 0.$$

By the above inequalities, we conclude that $x \in D(\overline{x})$. This means that any feasible solution in (VP) is also feasible in $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$. □

4. η-saddle point criteria for nonsmooth vector optimization

In this section, we use the η-approximation method to obtain new saddle point criteria for a class of nonsmooth multiobjective programming problems. In other words, we characterize solvability of nonconvex nonsmooth vector optimization problems with nondifferentiable V-invex functions (with respect to the same function η).

First, we introduce a definition of a family of the so-called η-Lagrange functions for the constructed η-approximated vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$ belonging to a family of vector optimization problems constructed for the considered nonsmooth multiobjective programming problem in the η-approximation method.

Definition 11. A family of the so-called η-approximated Lagrange functions is defined for the vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$ as follows

$$L_\eta\left(x, \lambda, \mu, \overline{\xi}, \overline{\zeta}\right) := \operatorname{diag} \lambda f(\overline{x}) + \mu^T g(\overline{x}) e + \left(\lambda^T \overline{\xi} + \mu^T \overline{\zeta}\right) \eta(x, \overline{x})\, e$$

$$:= \left(\lambda_1 f_1(\overline{x}) + \mu^T g(\overline{x}) + \left(\lambda^T \overline{\xi} + \mu^T \overline{\zeta}\right) \eta(x, \overline{x}), \ldots, \lambda_k f_k(\overline{x}) + \mu^T g(\overline{x}) + \left(\lambda^T \overline{\xi} + \mu^T \overline{\zeta}\right) \eta(x, \overline{x})\right),$$

where $\overline{\xi} = \left(\overline{\xi}_1, \ldots, \overline{\xi}_k\right),\ \overline{\zeta} = \left(\overline{\zeta}_1, \ldots, \overline{\zeta}_m\right),\ \overline{\xi}_i,\ i = 1, \ldots, k, \overline{\zeta}_j,\ j = 1, \ldots, m$, are Clarke's generalized gradients of $f_i,\ i = 1, \ldots, k$, and $g_j,\ j = 1, \ldots, m$, at $\overline{x}$, respectively, that is, $\overline{\xi}_i \in \partial f_i(\overline{x}),\ i = 1, \ldots, k, \overline{\zeta}_j \in \partial g_j(\overline{x}),\ j = 1, \ldots, m$, and η is a vector-valued function defined by $\eta : X \times X \to R^n$.

For the Lagrange function, some kinds of saddle points have been introduced (see, for example, [1,10]). Now, we give the following definition of a so-called η-saddle point for the η-Lagrange function in the η-approximated vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$.

Definition 12. A point $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right) \in D \times R^k_+ \times R^m_+$ is said to be an (Pareto) η-saddle point for the η-approximated Lagrange function defined for an η-approximated vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$ if,

(i) $L_\eta\left(\overline{x}, \overline{\lambda}, \mu, \overline{\xi}, \overline{\zeta}\right) \leqq L_\eta\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}, \overline{\zeta}\right)\ \forall \mu \in R^m_+$,
(ii) $L_\eta\left(x, \overline{\lambda}, \overline{\mu}, \overline{\xi}, \overline{\zeta}\right) \nleq L_\eta\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}, \overline{\zeta}\right)\ \forall x \in D$.

Remark 13. As it follows from the above definition of an (Pareto) η-saddle point for the η-approximated Lagrange function defined for an η-approximated vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$ and the definition of the η-approximated Lagrange function, the condition (i) in Definition 12 is equivalent, under assumption $\eta(\overline{x}, \overline{x}) = 0$, to the relation

$$\overline{\mu}^T g(\overline{x}) = 0. \tag{10}$$

Indeed, if $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right) \in D \times R^k_+ \times R^m_+$ is an (Pareto) η-saddle point for the η-approximated Lagrange function defined for an η-approximated vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$, then, by condition (i) in Definition 12, the following inequality

$$\operatorname{diag} \overline{\lambda} f(\overline{x}) + \mu^T g(\overline{x}) e + \left(\overline{\lambda}^T \overline{\xi} + \mu^T \overline{\zeta}\right) \eta(\overline{x}, \overline{x})\, e \leqq \operatorname{diag} \overline{\lambda} f(\overline{x}) + \overline{\mu}^T g(\overline{x}) e + \left(\overline{\lambda}^T \overline{\xi} + \overline{\mu}^T \overline{\zeta}\right) \eta(\overline{x}, \overline{x})\, e. \tag{11}$$

holds for all $\mu \in R^m_+$, any $\overline{\xi} = \left(\overline{\xi}_1, \ldots, \overline{\xi}_k\right)$, where $\overline{\xi}_i \in \partial f_i(\overline{x})$ and any $\overline{\zeta} = \left(\overline{\zeta}_1, \ldots, \overline{\zeta}_m\right)$, where $\overline{\zeta}_j \in \partial g_j(\overline{x})$. By assumption, $\eta(\overline{x}, \overline{x}) = 0$. Thus, we get, for all $\mu \in R^m_+$,

$$\mu^T g(\overline{x}) \leqq \overline{\mu}^T g(\overline{x}). \tag{12}$$

In (12), let $\mu = 0$. Hence,

$$\overline{\mu}^T g(\overline{x}) \geqq 0. \tag{13}$$

Using $\overline{x} \in D$ together with $\overline{\mu} \in R^m_+$, we obtain

$$\overline{\mu}^T g(\overline{x}) \leqq 0. \tag{14}$$

By (13) and (14), it follows that the relation (10) is satisfied. Further, it is not difficult to prove that if the relation (10) is satisfied, then the inequality (11) is also satisfied.

As it follows directly from the definition of the η-approximated Lagrange function defined for an η-approximated vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$, condition (ii) in Definition 12 is equivalent to:

$$\left(\overline{\lambda}^T\overline{\xi} + \overline{\mu}^T\overline{\zeta}\right)(\eta(x, \overline{x}) - \eta(\overline{x}, \overline{x}))\, e \nleq 0, \quad \forall x \in D. \tag{15}$$

Now, we prove the necessary condition for a point $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ to be an η-saddle point for the η-approximated Lagrange function defined for an η-approximated vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$.

Theorem 14. *Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ be an η-saddle point for L_η defined in an η-approximated vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$. Further, assume that f is nondifferentiable V-invex at $\overline{x}$ on D with respect to η satisfying the condition $\eta(\overline{x}, \overline{x}) = 0$ and the constraint function g_j, $j \in J(\overline{x})$, is locally Lipschitz β_j-invex at $\overline{x}$ on D with respect to the same function η. Then $\overline{x}$ is a weak Pareto solution in (VP).*

Proof. Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ be an η-saddle point for L_η. We proceed by contradiction. Let us suppose that $\overline{x}$ is not a weak Pareto solution in (VP). Then, there exists $\widetilde{x} \in D$ such that

$$f(\widetilde{x}) < f(\overline{x}). \tag{16}$$

By assumption, f is a nondifferentiable V-invex function at $\overline{x}$ on D with respect to η. Hence, by Definition 7, the following inequalities

$$f_i(x) - f_i(\overline{x}) \geqq \alpha_i(x, \overline{x})\, \overline{\xi}_i^T \eta(x, \overline{x}), \tag{17}$$

hold for all $x \in D$ and any $\overline{\xi}_i \in \partial f_i(\overline{x})$, $i \in I$. Therefore, it is also satisfied for $x = \widetilde{x}$. Thus, by (16) and (17), for any $i \in I$,

$$\alpha_i(\widetilde{x}, \overline{x})\, \overline{\xi}_i^T \eta(\widetilde{x}, \overline{x}) < 0. \tag{18}$$

By assumption, $\alpha_i(x, \overline{x}) > 0$, $i \in I$, for all $x \in D$. Hence, (18) gives

$$\overline{\xi}_i^T \eta(\widetilde{x}, \overline{x}) < 0. \tag{19}$$

Since $\overline{\lambda} \geq 0$, then (19) yields

$$\overline{\lambda}_i \overline{\xi}_i \eta(\widetilde{x}, \overline{x}) \leqq 0, \tag{20}$$

but at least for one $i \in I$,

$$\overline{\lambda}_i \overline{\xi}_i^T \eta(\widetilde{x}, \overline{x}) < 0. \tag{21}$$

Thus, by (20) and (21),

$$\overline{\lambda}\overline{\xi}\eta(\widetilde{x}, \overline{x})e \leq 0. \tag{22}$$

By assumption, the constraint function g_j, $j \in J(\overline{x})$, is locally Lipschitz β_j-invex with respect to η at $\overline{x}$ on D. Thus, by Definition 7, the following inequalities

$$g_j(x) - g_j(\overline{x}) \geqq \beta_j(x, \overline{x})\, \overline{\zeta}_j^T \eta(x, \overline{x}), \tag{23}$$

hold for all $x \in D$ and any $\overline{\zeta}_j \in \partial g_j(\overline{x})$, $j \in J$. Multiplying (23) by $\overline{\mu}_j \geqq 0$ and then using (10), we get

$$\overline{\mu}_j g_j(x) \geqq \beta_j(x, \overline{x})\, \overline{\mu}_j \overline{\zeta}_j \eta(x, \overline{x}). \tag{24}$$

Since $\overline{\mu}_j \geqq 0$ and $x \in D$, then the inequalities

$$0 \geqq \beta_j(x, \overline{x})\, \overline{\mu}_j \overline{\zeta}_j^T \eta(x, \overline{x}) \tag{25}$$

hold for all $x \in D$ and any $\overline{\zeta}_j \in \partial g_j(\overline{x})$, $j \in J$. By assumption, $\beta_j(x, \overline{x}) > 0$, $j \in J$, for all $x \in D$. Thus,

$$\overline{\mu}_j \overline{\zeta}_j^T \eta(x, \overline{x}) \leqq 0. \tag{26}$$

Adding both sides of the above inequalities, we obtain that the inequality

$$\overline{\mu}^T \overline{\zeta} \eta(x, \overline{x}) e \leqq 0 \tag{27}$$

holds for all $x \in D$. Thus, it is also satisfied for $x = \widetilde{x}$. Combining (22) and (27), we get that the inequality

$$\left(\overline{\lambda}^T \overline{\xi} + \overline{\mu}^T \overline{\zeta}\right) \eta(\widetilde{x}, \overline{x}) e \leq 0 \tag{28}$$

holds. By assumption, $\eta(\overline{x}, \overline{x}) = 0$. Therefore,

$$\left(\overline{\lambda}^T \overline{\xi} + \overline{\mu}^T \overline{\zeta}\right) \eta(\overline{x}, \overline{x}) e = 0. \tag{29}$$

By (28) and (29), the following inequality

$$\left(\overline{\lambda}^T \overline{\xi} + \overline{\mu}^T \overline{\zeta}\right) \left(\eta(\widetilde{x}, \overline{x}) - \eta(\overline{x}, \overline{x})\right) e \leq 0$$

holds, contradicting (15) (see Remark 13). Thus $\overline{x}$ is a weak Pareto optimal solution in the original nonsmooth multiobjective programming problem (VP). □

Theorem 15. *Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ be an η-saddle point for L_η defined in the η-approximated vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$. Further, assume that f is strictly V-invex at $\overline{x}$ on D with respect to η satisfying the condition $\eta(\overline{x}, \overline{x}) = 0$ and the constraint function g_j, $j \in J(\overline{x})$, is V-invex at $\overline{x}$ on D with respect to the same function η. If the Lagrange multiplier $\overline{\lambda} > 0$, then $\overline{x}$ is a Pareto optimal solution in (VP).*

Now, we prove the above result under the Lagrangian assumption.

Theorem 16. *Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ be an η-saddle point for L_η defined in the η-approximated vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$. Further, assume that f and g are regular in the sense of Clarke [17] at $\overline{x}$ and, moreover, one of the following hypotheses is satisfied:*

(a) *L is V-invex at $\overline{x}$ on D with respect to η and the Lagrange multiplier is assumed to satisfy $\overline{\lambda} > 0$,*
(b) *L is strictly V-invex at $\overline{x}$ on D with respect to η.*

If the function η satisfies the condition $\eta(\overline{x}, \overline{x}) = 0$, then $\overline{x}$ is a Pareto optimal solution in the original nonsmooth multiobjective programming problem (VP).

Proof. We proceed by contradiction. Suppose that $\overline{x}$ is not a Pareto optimal solution in the original nonsmooth multiobjective programming problem (VP). Then, there exists $\widetilde{x} \in D$ such that

$$f(\widetilde{x}) \leq f(\overline{x}). \tag{30}$$

From the assumption, the Lagrange multiplier $\overline{\lambda} > 0$. Thus, (30) yields

$$\operatorname{diag} \overline{\lambda} f(\widetilde{x}) \leq \operatorname{diag} \overline{\lambda} f(\overline{x}). \tag{31}$$

By assumption, $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ is an η-saddle point for L_η. Hence, by Remark 13, it follows that

$$\overline{\mu}^T g(\overline{x}) = 0. \tag{32}$$

Therefore, from the feasibility of $\widetilde{x}$ in (VP), it follows that

$$\overline{\mu}^T g(\widetilde{x}) \leqq \overline{\mu}^T g(\overline{x}). \tag{33}$$

By assumption, L is V-invex at $\overline{x}$ on D with respect to η. Then, by Definition 7, it follows that

$$L_i\left(x, \overline{\lambda}, \overline{\mu}\right) - L_i\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right) \geqq \gamma_i(x, \overline{x}) \overline{\vartheta}_i^T \eta(x, \overline{x}) \tag{34}$$

for all $x \in D$ and any $\overline{\vartheta}_i \in \partial L_i\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$. Thus, it is also satisfied for $x = \widetilde{x}$. By assumption, f and g are regular at $\overline{x}$. Then, by Corollaries 2 and 3 for Proposition 2.3.3 [17], we have

$$\partial L_i\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right) = \partial\left(\overline{\lambda}_i f_i + \frac{1}{k}\overline{\mu} g(\overline{x})\right) = \overline{\lambda}_i \partial f_i(\overline{x}) + \frac{1}{k}\partial\left(\overline{\mu}^T g(\overline{x})\right) = \overline{\lambda}_i \partial f_i(\overline{x}) + \frac{1}{k}\sum_{j=1}^{m} \overline{\mu}_j \partial g_j(\overline{x}). \tag{35}$$

Using the definition of the Lagrange function L (see (3)) together with (34) and (35), we obtain that, for each $i \in I$,

$$\overline{\lambda}_i f_i(\widetilde{x}) + \overline{\mu}^T g(\widetilde{x}) - \left(\overline{\lambda}_i f_i(\overline{x}) + \overline{\mu}^T g(\overline{x})\right) \geqq \gamma_i(\widetilde{x}, \overline{x}) \left(\overline{\lambda}_i \overline{\xi}_i + \frac{1}{k}\overline{\mu}^T \overline{\zeta}\right) \eta(\widetilde{x}, \overline{x}), \tag{36}$$

where $\overline{\xi} = \left(\overline{\xi}_1, \ldots, \overline{\xi}_k\right)$ and $\overline{\xi}_i,\ i = 1, \ldots, k$, are Clarke's generalized gradients of $f_i,\ i = 1, \ldots, k$, at $\overline{x}$, that is, $\overline{\xi}_i \in \partial f_i(\overline{x}),\ i = 1, \ldots, k, \overline{\zeta} = \left(\overline{\zeta}_1, \ldots, \overline{\zeta}_m\right)$ and $\overline{\zeta}_j,\ j = 1, \ldots, m$, are Clarke's generalized gradients of $g_j,\ j = 1, \ldots, m$, at $\overline{x}$, that is, $\overline{\zeta}_j \in \partial g_j(\overline{x}),\ j = 1, \ldots, m$. By (31), (33) and (36), for each $i \in I$,

$$0 \geqq \gamma_i\left(\widetilde{x}, \overline{x}\right)\left(\overline{\lambda}_i \overline{\xi}_i + \frac{1}{k}\overline{\mu}^T \overline{\zeta}\right)\eta\left(\widetilde{x}, \overline{x}\right), \tag{37}$$

but at least for one $i \in I$,

$$0 > \gamma_i\left(\widetilde{x}, \overline{x}\right)\left(\overline{\lambda}_i \overline{\xi}_i + \frac{1}{k}\overline{\mu}^T \overline{\zeta}\right)\eta\left(\widetilde{x}, \overline{x}\right). \tag{38}$$

From the definition, $\gamma_i\left(\widetilde{x}, \overline{x}\right) > 0, i \in I$. Then, by (37) and (38),

$$0 \geq \left(\overline{\lambda}^T \overline{\xi} + \overline{\mu}^T \overline{\zeta}\right)\eta\left(\widetilde{x}, \overline{x}\right) e. \tag{39}$$

By assumption, $\eta\left(\overline{x}, \overline{x}\right) = 0$. Thus, the following relation

$$\left(\overline{\lambda}^T \overline{\xi} + \overline{\mu}^T \overline{\zeta}\right)\left(\eta\left(\widetilde{x}, \overline{x}\right) - \eta\left(\overline{x}, \overline{x}\right)\right) e \leq 0$$

is satisfied. This inequality is a contradiction to (15) (see Remark 13). Thus, the conclusion of this theorem is established. □

Now, we establish a converse condition, that is, a sufficient condition for a point $\left(\overline{x}, \overline{\lambda}, \overline{\xi}\right) \in D \times R^k_+ \times R^m_+$ to be an η-saddle point for the η-Lagrange function in an η-approximated vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$. To prove this result, we don't need to assume that the functions constituting the considered multiobjective programming problem (VP) are V-invex. Further, as it follows from this theorem, it turns out that not all of η-approximated vector optimization problems $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$ are equivalent to the original multiobjective programming problem in the sense of discussed in the paper.

Theorem 17. *Let $\overline{x}$ be a feasible solution in the original multiobjective programming problem (VP), at which the Karush–Kuhn–Tucker necessary optimality conditions (4)–(6) are satisfied with Lagrange multipliers $\overline{\lambda} \in R^k_+$ and $\overline{\mu} \in R^m_+$. Further, assume that the function η satisfies the following condition $\eta\left(\overline{x}, \overline{x}\right) = 0$. Then $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ is an η-saddle point for the η-Lagrange function in the η-approximated vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$, where $\overline{\xi} = \left(\overline{\xi}_1, \ldots, \overline{\xi}_k\right)$, $\overline{\zeta} = \left(\overline{\zeta}_1, \ldots, \overline{\zeta}_m\right)$, $\overline{\xi}_i,\ i = 1, \ldots, k, \overline{\zeta}_j, j = 1, \ldots, m$, are Clarke's generalized gradients of $f_i,\ i = 1, \ldots, k$, and $g_j, j = 1, \ldots, m$, at $\overline{x}$, respectively, satisfying the Karush–Kuhn–Tucker necessary optimality conditions (4)–(6) with the Lagrange multipliers $\overline{\lambda}$ and $\overline{\mu}$.*

Proof. By assumption, $\overline{x}$ is such a feasible point in the original multiobjective programming problem (VP), at which the Karush–Kuhn–Tucker necessary optimality conditions (4)–(6) are satisfied with Lagrange multipliers $\overline{\lambda} \in R^k_+$ and $\overline{\mu} \in R^m_+$. Then, by the Karush–Kuhn–Tucker necessary optimality condition (5), it follows that the inequality

$$\overline{\mu}^T g\left(\overline{x}\right) \geqq \mu^T g\left(\overline{x}\right)$$

holds for all $\mu \in R^m_+$. By assumption, $\eta\left(\overline{x}, \overline{x}\right) = 0$. Therefore,

$$\operatorname{diag} \overline{\lambda} f\left(\overline{x}\right) + \mu^T g\left(\overline{x}\right) e + \left(\overline{\lambda}^T \overline{\xi} + \mu^T \overline{\zeta}\right)\eta\left(\overline{x}, \overline{x}\right) e \leqq \operatorname{diag} \overline{\lambda} f\left(\overline{x}\right) + \overline{\mu}^T g\left(\overline{x}\right) e + \left(\overline{\lambda}^T \overline{\xi} + \overline{\mu}^T \overline{\zeta}\right)\eta\left(\overline{x}, \overline{x}\right) e.$$

Hence, it follows from Definition 11 that the inequality

$$L_\eta\left(\overline{x}, \overline{\lambda}, \mu, \overline{\xi}, \overline{\zeta}\right) \leqq L_\eta\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}, \overline{\zeta}\right) \tag{40}$$

is satisfied for all $\mu \in R^m_+$. This means that the inequality (i) from Definition 12 is established.

We now prove the second inequality in Definition 12. By the definition of the η-Lagrange function, we have

$$L_\eta\left(x, \overline{\lambda}, \overline{\mu}, \overline{\xi}, \overline{\zeta}\right) = \operatorname{diag} \overline{\lambda} f\left(\overline{x}\right) + \overline{\mu} g\left(\overline{x}\right) e + \left(\overline{\lambda \xi} + \overline{\mu \zeta}\right)\eta\left(x, \overline{x}\right) e$$

and

$$L_\eta\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}, \overline{\zeta}\right) = \operatorname{diag} \overline{\lambda} f\left(\overline{x}\right) + \overline{\xi} g\left(\overline{x}\right) e + \left(\overline{\lambda \xi} + \overline{\mu \zeta}\right)\eta\left(\overline{x}, \overline{x}\right) e.$$

Using the Karush–Kuhn–Tucker optimality conditions (4)–(5) together with the condition $\eta\left(\overline{x}, \overline{x}\right) = 0$, it follows that the relation

$$L_\eta\left(x, \overline{\lambda}, \overline{\mu}, \overline{\xi}, \overline{\zeta}\right) \nleq L_\eta\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}, \overline{\zeta}\right) \tag{41}$$

holds for all $x \in D$.

We conclude, by (40), (41) and Definition 12, that $\left(\overline{x}, \overline{\lambda}, \overline{\xi}\right)$ is an η-saddle point of the η-Lagrange function in an η-approximated vector optimization problem $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$ associated with the original multiobjective programming problem (VP). □

Remark 18. As it follows from Theorem 17, if $\bar{x}$ is a (weak) Pareto optimal solution in the original multiobjective programming problem (VP), then $\left(\bar{x}, \bar{\lambda}, \bar{\mu}\right)$ is an η-saddle point for the η-Lagrange function in such associated η-approximated vector optimization problems $(VP_\eta(\bar{x}, \bar{\xi}, \bar{\zeta}))$, where $\bar{\xi} = \left(\bar{\xi}_1, \ldots, \bar{\xi}_k\right)$, $\bar{\zeta} = \left(\bar{\zeta}_1, \ldots, \bar{\zeta}_m\right)$, and $\bar{\xi}_i,\ i = 1, \ldots, k, \bar{\zeta}_j,\ j = 1, \ldots, m$, are Clarke's generalized gradients of $f_i,\ i = 1, \ldots, k$, and $g_j,\ j = 1, \ldots, m$, at $\bar{x}$, respectively, satisfying the Karush–Kuhn–Tucker necessary optimality conditions (4)–(6) with the Lagrange multipliers $\bar{\lambda} \in R^k_+$ and $\bar{\mu} \in R^m_+$.

The following corollary follows directly from Theorem 17.

Corollary 19. *Let $\bar{x}$ be a (weak) Pareto optimal solution in the original multiobjective programming problem (VP). Further, assume that the hypotheses of Theorem 17 are satisfied. Then $\left(\bar{x}, \bar{\lambda}, \bar{\mu}\right)$ is an η-saddle point of the η-Lagrange function in the η-approximated vector optimization problem $(VP_\eta(\bar{x}, \bar{\xi}, \bar{\zeta}))$, where $\bar{\xi} = \left(\bar{\xi}_1, \ldots, \bar{\xi}_k\right)$, $\bar{\zeta} = \left(\bar{\zeta}_1, \ldots, \bar{\zeta}_m\right)$, $\bar{\xi}_i,\ i = 1, \ldots, k, \bar{\zeta}_j,\ j = 1, \ldots, m$, are Clarke's generalized gradients of $f_i,\ i = 1, \ldots, k$, and $g_j,\ j = 1, \ldots, m$, at $\bar{x}$, respectively, satisfying the Karush–Kuhn–Tucker necessary optimality conditions (4)–(6) with the Lagrange multipliers $\bar{\lambda}$ and $\bar{\mu}$.*

In view of Theorem 14 and Corollary 19, we see that, if we assume that f is (V-invex) strictly V-invex with respect to η and α and g is V-invex at $\bar{x}$ on D with respect to the same function η and β, but not necessarily β is equal to α, η satisfies the condition $\eta(\bar{x}, \bar{x}) = 0$, and, moreover, some constraint qualification is satisfied at $\bar{x}$, then the η-approximation approach guarantees the equivalence between a (weak) Pareto solution $\bar{x}$ in (VP) and an η-saddle point of the η-Lagrange function in its associated η-approximated vector optimization problem $(VP_\eta(\bar{x}, \bar{\xi}, \bar{\zeta}))$ (where $\bar{\xi} = (\bar{\xi}_1, \ldots, \bar{\xi}_k)$, $\bar{\zeta} = (\bar{\zeta}_1, \ldots, \bar{\zeta}_m)$) and $\bar{\xi}_i,\ i = 1, \ldots, k,\ \bar{\zeta}_j, j = 1, \ldots, m$, are Clarke's generalized gradients of $f_i,\ i = 1, \ldots, k$, and $g_j, j = 1, \ldots, m$, at $\bar{x}$, respectively, satisfying the Karush–Kuhn–Tucker necessary optimality conditions (4)–(6) with the Lagrange multipliers $\bar{\lambda} \in R^k_+$ and $\bar{\mu} \in R^m_+$) in the sense discussed above.

Now, we give an example of a multiobjective programming problem $(VP1)$ which, by using the approach discussed in this paper, is transformed to less complicated vector optimization problems $(VP1_\eta\ (\bar{x}, \bar{\xi}, \bar{\zeta}))$. For the considered multiobjective programming problem, we show the equivalence between its Pareto optimal solution $\bar{x}$ and an η-saddle point of the η-Lagrange function in its associated η-approximated vector optimization problems $(VP1_\eta\ (\bar{x}, \bar{\xi}, \bar{\zeta}))$.

Example 20. We consider the following multiobjective programming problem

$$\begin{aligned} &V\text{-Minimize}\, f(x) = \left(e^{x^2+|x|+1}, \arctan|x| + 1\right) \\ &g(x) = 1 - e^x \leqq 0. \end{aligned} \tag{$VP1$}$$

Note that $D = \{x \in R : x \geqq 0\}$ and $\bar{x} = 0$ is a Pareto optimal point in the considered multiobjective programming problem. It is not difficult to prove that f is strictly V-invex at $\bar{x}$ on D with respect to $\eta^{(1)}$ and with respect to $\alpha^{(1)} = \left(\alpha_1^{(1)}, \alpha_2^{(1)}\right)$ and g is strictly V-invex at $\bar{x}$ on D with respect to the same function $\eta^{(1)}$ and with respect to $\beta^{(1)}$, where

$$\eta^{(1)}(x, \bar{x}) = x - \bar{x}, \tag{42}$$

$$\alpha_1^{(1)}(x, \bar{x}) = \begin{cases} \dfrac{e^{x^2+|x|} - 1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases} \qquad \alpha_2^{(1)}(x, \bar{x}) = \begin{cases} \dfrac{\arctan|x|}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases}$$

$$\beta^{(1)}(x, \bar{x}) = \begin{cases} \dfrac{e^x - 1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Now, using the approach discussed in the paper, we construct a family of associated η-approximated vector optimization problems $VP1_{\eta^{(1)}}(\bar{x}, \bar{\xi}, \bar{\zeta})$. Therefore, both the objective function f and the constraint function g are η-approximated at $\bar{x}$. Thus, we obtain the following family of linear vector optimization problems

$$\begin{aligned} &V\text{-Minimize}\ \left(e + \bar{\xi}_1 x,\ 1 + \bar{\xi}_2 x\right) \\ &-x \leqq 0, \end{aligned} \tag{$VP1_{\eta^{(1)}}(\bar{x}, \bar{\xi}, \bar{\zeta})$}$$

where $\bar{\xi}_i \in \partial f_i(\bar{x})$, $i = 1, 2$ and, moreover, $\bar{\lambda}_1\bar{\xi}_1 + \bar{\lambda}_2\bar{\xi}_2 - \bar{\mu} = 0$, where $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\mu}$ are Lagrange multipliers satisfying the Karush–Kuhn–Tucker necessary optimality conditions. It is not difficult to see that, similarly as in the original multiobjective programming problem, $\bar{x} = 0$ is also a Pareto optimal solution in the above family of η-approximated vector optimization problems $(VP1_{\eta^{(1)}}(\bar{x}, \bar{\xi}, \bar{\zeta}))$. We now define the η-Lagrange function in $(VP1_{\eta^{(1)}}(\bar{x}, \bar{\xi}, \bar{\zeta}))$. Then, by Definition 11, we have

$$L_{\eta^{(1)}}\left(x, \lambda, \mu, \bar{\xi}, -1\right) = (\lambda_1 e + (\lambda_1\xi_1 + \lambda_2\xi_2 - \mu)\, x,\ \lambda_2 + (\lambda_1\xi_1 + \lambda_2\xi_2 - \mu)\, x)\,.$$

It is not difficult to prove, by Definition 12, that $\left(\bar{x}, \bar{\lambda}, \bar{\mu}\right) = \left(0, (\bar{\lambda}_1, \bar{\lambda}_2), \bar{\mu}\right)$, where $\bar{\lambda}_1\bar{\xi}_1 + \bar{\lambda}_2\bar{\xi}_2 = \bar{\mu}$, is an η-saddle point in the family of η-approximated vector optimization problems $(VP1_{\eta^{(1)}}(\bar{x}, \bar{\xi}, \bar{\zeta}))$ constructed in the η-approximation method. Since all hypotheses of Theorem 14 are satisfied, then $\bar{x} = 0$ is Pareto optimal in the original considered multiobjective programming problem. Thus, we establish the equivalence between a Pareto optimal solution $\bar{x} = 0$ in the considered original multiobjective programming problem and an η-saddle point $\left(\bar{x}, \bar{\lambda}, \bar{\xi}\right)$ in its associated η-approximated vector optimization problems $(VP1_{\eta^{(1)}}(\bar{x}, \bar{\xi}, \bar{\zeta}))$. It is not difficult to see that the functions constituting the considered multiobjective programming problem $(VP1)$ are not invex with respect to $\eta^{(1)}$ defined above. Therefore, we cannot use the results established in [16]. In other words, the $\eta^{(1)}$-approximation method is not applicable to this vector optimization problem under invexity assumption. It is not difficult to see that the function $\eta^{(1)}$ has a simpler form under assumption V-invexity (it is a linear function with respect to the first component) in comparison to each function η with respect to which these functions are invex.

Remark 21. Further, we define the classical vector-valued Lagrange function for the original multiobjective programming problem considered in Example 20. Thus, by (3),

$$L(x, \lambda, \mu) = \left(\lambda_1 e^{x^2+|x|+1} + \mu\left(1 - e^x\right), \ \lambda_2\left(\arctan|x| + 1\right) + \mu\left(1 - e^x\right)\right).$$

Now, it is not difficult to note, comparing the forms of the classical vector-valued Lagrange function and the η-Lagrange function, that the first one from them is more complicated than the second one. Therefore, it is easier to solve the η-saddle point criteria for the η-approximated vector optimization problem (since they are defined by using the η-Lagrange function) than the classical saddle point criteria defined for the originally multiobjective programming problem (since they are formulated by the help of the vector-valued Lagrange function). As it follows from the considered example, under nondifferentiable V-invexity assumption imposed on the functions constituting $(VP1)$, the η-Lagrange function is linear with respect to the first component for some class of nonlinear vector optimization problems. In this way, solvability of nonlinear nonconvex multiobjective programming problems can be characterized by the help of the modified saddle point criteria defined for their associated linear vector optimization problems. This property is useful from the practical point of view.

Remark 22. Note that, in general, there exists more than one function η with respect to which all functions involved in the original multiobjective programming problem are V-invex. This means that, in general, there exists more than one family of η-approximated vector optimization problems associated with the original multiobjective programming problem. Indeed, it is not difficult to prove that the functions constituting nonsmooth optimization problem in Example 20 are also strictly V-invex at $\bar{x}$ on D with respect to the following function $\eta^{(2)}$ defined by

$$\eta^{(2)}(x, \bar{x}) = e^x - e^{\bar{x}}, \tag{43}$$

where

$$\alpha_1^{(2)}(x, \bar{x}) = \frac{1}{x^2+1}, \qquad \alpha_2^{(2)}(x, \bar{x}) = \begin{cases} \dfrac{\arctan|x|}{e^x - 1} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases}$$

$$\beta^{(2)}(x, \bar{x}) \geqq 1 \quad \forall x \in D.$$

Then, we obtain the following family of vector η-approximation optimization problems

$$\begin{aligned} &V\text{-Minimize } \left(e + \bar{\xi}_1\left(e^x - 1\right), \ 1 + \bar{\xi}_2\left(e^x - 1\right)\right) \\ &1 - e^x \leqq 0. \end{aligned} \tag{$VP1_{\eta^{(2)}}(\bar{x}, \bar{\xi}, \bar{\zeta})$}$$

Of course, this is not a family of linear vector optimization problems. This follows from the fact that the function $\eta^{(2)}$, with respect to which the functions constituting the considered nonsmooth vector optimization problem $(VP1)$ are nondifferentiable V-invex, is not linear with respect to the first component. However, we obtain a family of smooth vector optimization problems in two considered cases of the functions η. This means that we are in position to characterize solvability of a nonsmooth nonconvex vector optimization problem (VP) by the help of a family of smooth vector optimization problems, that is, a family of (smooth) vector η-approximated optimization problems. What is interesting, in most cases, one of such families of smooth vector optimization problems is a family of linear vector optimization problems. This property is valid from the practical point of view.

Remark 23. As it follows even from Example 20, the less complicated function η, with respect to which all functions constituting the original multiobjective programming problem are V-invex, should be used to construct η-approximated vector optimization problems. This is a consequence of the fact that then η-approximated vector optimization problems have a less complicated form in comparison to the original multiobjective programming problem. In the case, when a function η is linear with respect to the first component, we obtain a family of linear vector optimization problems as one of families of η-approximated vector optimization problems associated with the original multiobjective programming problem. Also this property is useful from the practical point of view. In this case, solvability of a nonlinear nonconvex multiobjective programming problem is characterized by the help of a family of linear vector optimization problems.

The assumption that a function η satisfies the condition $\eta(\overline{x}, \overline{x}) = 0$ is essential to confirm the equivalence between the vector optimization problems (VP) and $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$ in the sense discussed in the paper. If this condition is not satisfied, then there is no the equivalence between the original multiobjective programming problem (VP) and none of the associated η-approximated vector optimization problems belonging to a family of problems $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$ in the sense discussed in the paper. Now, we give an example of such a nonsmooth vector optimization problem.

Example 24. We consider the following nonsmooth multiobjective programming problem

$$\begin{aligned} &V\text{-Minimize}\, f(x) = \left(|x_1| + x_2,\ \ \arctan(x_1) + x_2^2 + 1\right) \\ &g_1(x) = x_1^2 + x_2 \leqq 0, \\ &g_2(x) = x_2^2 - 1 \leqq 0. \end{aligned} \tag{VP2}$$

Note that $D = \left\{(x_1, x_2) \in R^2 : x_1^2 + x_2 \leqq 0 \wedge -1 \leqq x_2 \leqq 0\right\}$ and $\overline{x} = (0, 0)$ is a Pareto optimal point in the considered multiobjective programming problem (VP2). It is not difficult to prove that f is strictly V-invex at $\overline{x}$ on D with respect to η and with respect to $\alpha = (\alpha_1, \alpha_2)$ and g is strictly V-invex at $\overline{x}$ on D with respect to the same function η and with respect to $\beta = (\beta_1, \beta_2)$, where

$$\begin{aligned} &\eta(x, \overline{x}) = \begin{bmatrix} \eta_1(x, \overline{x}) \\ \eta_2(x, \overline{x}) \end{bmatrix}, \\ &\eta_1(x, \overline{x}) = x_1 - 2, \qquad \eta_2(x, \overline{x}) = x_2 - 2 - x_1^2, \\ &\alpha_1(x, \overline{x}) = 1, \qquad \alpha_2(x, \overline{x}) = \frac{1}{x_1 + 2}, \qquad \beta_1(x, \overline{x}) = 1, \qquad \beta_2(x, \overline{x}) = 1. \end{aligned}$$

It is not difficult to see that the function η defined above does not satisfy the condition $\eta(\overline{x}, \overline{x}) = 0$. However, for the considered multiobjective programming problem (VP2), we construct a family of its vector η-approximated optimization problems $(VP2_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$, where $\overline{\xi} = \left(\overline{\xi}_1, \ldots, \overline{\xi}_k\right)$, $\overline{\zeta} = \left(\overline{\zeta}_1, \ldots, \overline{\zeta}_m\right)$ and $\overline{\xi}_i$, $i = 1, \ldots, k$, $\overline{\zeta}_j$, $j = 1, \ldots, m$, are Clarke's generalized gradients of f_i, $i = 1, 2$, and g_j, $j = 1, 2$, at $\overline{x}$ respectively, that is, $\overline{\xi}_i \in \partial f_i(\overline{x})$, $i = 1, 2$, $\overline{\zeta}_j \in \partial g_j(\overline{x})$, $j = 1, 2$. Then, we obtain the following family of vector optimization problems

$$\begin{aligned} &V\text{-Minimize}\ \left(\overline{\xi}_1^1 (x_1 - 2) + x_2 - 2 - x_1^2,\ x_1 - 1 + \overline{\xi}_2^2 \left(x_2 - 2 - x_1^2\right)\right) \\ &x_2 - 2 - x_1^2 \leqq 0. \end{aligned} \tag{$VP2_\eta(\overline{x}, \overline{\xi}, \overline{\zeta})$}$$

It is not difficult to see that $\overline{x} = (0, 0)$ is not a Pareto optimal point in any considered multiobjective programming problems $(VP2_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$. This follows from the fact that the set of all feasible solutions in η-approximated vector optimization problem belonging to a family of vector optimization problems $(VP2_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$ is unbounded. Therefore, the condition $\eta(\overline{x}, \overline{x}) = 0$ is essential to prove the equivalence between the original nonsmooth multiobjective programming problem and its associated η-approximated vector optimization problems belonging to a family of vector optimization problems $(VP2_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$ constructed in the η-approximation approach.

5. Concluding remarks

New saddle point criteria have been introduced using the η-approximation method to characterize solvability a class of nonconvex nondifferentiable multiobjective programming problems. In this approach, for the given function η, a family of the so-called η-approximated vector optimization problems is constructed for the original nondifferentiable multiobjective programming problem in the opposite to the case of differentiable multiobjective programming problems (see [14]), in which only one η-approximated vector optimization problem is constructed for the given function η. As it follows from the formulation of an η-approximated vector optimization problem, we need Lagrange multipliers of the original multiobjective programming problem to construct this modified vector optimization problem. This is a consequence of the fact, that we need a feasible point $\overline{x}$ in the original multiobjective programming problem, which is suspected to be optimal. Further, the equivalence between of a (weak) Pareto optimal solution in the original nondifferentiable multiobjective programming problem (VP) and an η-saddle point $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ of the η-Lagrange function of its η-approximated vector optimization problems $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$ is established only for such vector optimization problems $(VP_\eta(\overline{x}, \overline{\xi}, \overline{\zeta}))$, where $\overline{\xi} = \left(\overline{\xi}_1, \ldots, \overline{\xi}_k\right)$, $\overline{\zeta} = \left(\overline{\zeta}_1, \ldots, \overline{\zeta}_m\right)$, for which $\overline{\xi}_i$, $i = 1, \ldots, k$, $\overline{\zeta}_j$, $j = 1, \ldots, m$, Clarke's generalized gradients of f_i, $i = 1, \ldots, k$, and of g_j, $j = 1, \ldots, m$, at $\overline{x}$, respectively, satisfy the Karush–Kuhn–Tucker necessary optimality conditions (4)–(6) with the Lagrange multipliers $\overline{\lambda} \in R_+^k$ and $\overline{\mu} \in R_+^m$. In general, we obtain simpler vector optimization problems to solve than the original nonlinear nondifferentiable multiobjective programming problem. Moreover, there may exist more than one family of associated η-approximated vector optimization problems. In most cases, these are families of differentiable (linear) vector optimization problems. These properties are also useful from the practical point of view. It turns out that these properties are still valid for a class of nonsmooth vector optimization problems with nondifferentiable V-invex functions with respect to the same function η.

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